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THEORIES WITH ONLY A FINITE NUMBER OF EXISTENTIALLY COMPLETE MODELS

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ABSTRACT
We construct, in particular, a countable universal theory with JEP which has exactly 2 non-isomorphic countable existentially complete models, and these two models can be either elementarily equivalent or inequivalent.

The theory of countable universal theories with JEP (the joint-embedding property) and their countable existentially complete models, in some aspects of its development, is analogous to the theory of countable complete theories and their countable models. This analogy is systematically exposed in [5]. Indeed, by pursuing this analogy we were led to the simple examples in [4] of countable universal theories T_n, for $2 \leqq n < \omega$, with JEP which have exactly $n+1$ elementary equivalence classes of existentially complete models. This complemented the example in [1] of a theory T with JEP which has exactly 2 elementary equivalence classes of existentially complete models. One difference between these examples is that each of the T_n has exactly $n+1$ non-isomorphic countable existentially complete models, whereas T has infinitely many non-isomorphic countable existentially complete models. It appeared that there was some obstruction to extending our examples so as to get a theory T_1 with exactly 2 non-isomorphic countable existentially complete models. Was this apparent obstruction somehow connected to an analogue of Vaught's remarkable result [6] that a complete theory cannot have exactly 2 non-isomorphic countable models? At the time we wrote [4] we felt that this was so, seemingly not a completely heretical belief since Simmons [5] conjectured that there indeed was such an analogue of Vaught's theorem.

In this note, however, we construct examples of theories which eliminate any hope of finding such an analogue. In particular, we construct a countable universal theory with JEP which has exactly 2 non-isomorphic countable

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existentially complete models, and these two models can be chosen to be either elementarily equivalent or not.[†]

Suppose that $(P, \wedge, \vee, \leqq)$ is a finite lattice. Each subset $Q \subseteq P$ induces an equivalence relation on P in which two elements $x, y \in P$ are *Q-equivalent* iff for each $q \in Q$, $q \leqq x$ iff $q \leqq y$.

THEOREM. *Suppose $(P, \wedge, \vee, \leqq)$ is a finite lattice, and $Q \subseteq P$. Then there is a countable universal theory T with JEP, and there are $\mathfrak{A}_p$ for each $p \in P$, such that:*

(1) *$\mathfrak{B}$ is a countable existentially complete model of T iff $\mathfrak{B} \cong \mathfrak{A}_p$ for some $p \in P$;*

(2) *$\mathfrak{A}_p$ is embeddable in $\mathfrak{A}_q$ iff $p \leqq q$;*

(3) *$\mathfrak{A}_p \equiv \mathfrak{A}_q$ iff p and q are Q-equivalent.*

PROOF. Let us assume at the outset that $|P| > 1$. For, if $|P| = 1$ then we can get many examples of such T by using the result of Saracino [3] and letting T be the universal part of any $\aleph_0$-categorical theory.

It will be much more convenient at first to consider the notion of positive existential completeness, rather than just existential completeness. For this we need a weakening of the notion of embedding. We say that $\mathfrak{A}$ is *positively embedded* in $\mathfrak{B}$ iff $A \subseteq B$ and whenever R is an r-ary relation symbol and $a_0, \cdots, a_{r-1} \in A$ are such that $\mathfrak{A} \models R(a_0, \cdots, a_{r-1})$, then $\mathfrak{B} \models R(a_0, \cdots, a_{r-1})$. In this case we write $\mathfrak{A} \subseteq_p \mathfrak{B}$. We say that $\mathfrak{A}$ is *positively embeddable* in $\mathfrak{B}$ iff there is $\mathfrak{B}_1$ such that $\mathfrak{A} \subseteq_p \mathfrak{B}_1 \cong \mathfrak{B}$; and if $f : \mathfrak{B}_1 \to \mathfrak{B}$ is the isomorphism, then $f \restriction A$ is a *positive embedding* of $\mathfrak{A}$ into $\mathfrak{B}$. A theory T has the *positive joint-embedding property* (PJEP) iff for each two of its models, there is a third in which both are positively embeddable. By the usual diagram argument it can be shown that T has PJEP iff T can be extended to a complete theory which has the same negative universal sentences as T. (A sentence is negative if it is (logically equivalent to) the negation of a positive sentence. We allow that in a positive sentence both $=$ and $\neq$ can occur.) If T has PJEP, then a model $\mathfrak{A}$ of T is *positively existentially complete* (p.e.c.) if whenever $a_0, \cdots, a_{n-1} \in A$, $\phi(x_0, \cdots, x_{n-1})$ is a positive existential formula, $\mathfrak{A} \subseteq_p \mathfrak{B}$, $\mathfrak{B}$ is a model of T, and $\mathfrak{B} \models \phi(a_0, \cdots, a_{n-1})$, then $\mathfrak{A} \models \phi(a_0, \cdots, a_{n-1})$.

We will now prove the Theorem in the context of the above definitions.

THEOREM (positive version). *Suppose $(P, \wedge, \vee, \leqq)$ is a finite lattice, and $Q \subseteq P$. Then there is a countable negative universal theory T_Q with PJEP, and there are $\mathfrak{A}_p$ for each $p \in P$ such that:*

[†] *Added in proof.* J. Hirschfeld has also succeeded in producing a theory with JEP which has exactly 2 non-isomorphic countable existentially complete models. This example can be found in his paper, *Examples in the theory of existential completeness.*

(1) $\mathfrak{B}$ *is a countable p.e.c. model of* T *iff* $\mathfrak{B} \cong \mathfrak{A}_p$ *for some* $p \in P$;

(2) $\mathfrak{A}_p$ *is positively embeddable in* $\mathfrak{A}_q$ *iff* $p \leqq q$;

(3) $\mathfrak{A}_p \equiv \mathfrak{A}_q$ *iff* p *and* q *are* Q*-equivalent.*

We will first consider the case $Q = 0$, and then later show what changes to make for arbitrary $Q \subseteq P$.

Let $P = \{p_0, \cdots, p_n\}$, where p_n is the least element of P, and where $p_i \neq p_j$ whenever $i < j \leqq n$. Let H be the set of all triples $\langle c, d, e \rangle$ where $c, d, e < n$ and $p_e \leqq p_c \vee p_d$.

We now let L_0 be the language consisting of the following symbols:

(1) a unary relation symbol U^k for each $k < n$;

(2) a unary relation symbol U_i^k for each $k < n$ and $i < \omega$;

(3) a ternary relation symbol R_{cde} for each $\langle c, d, e \rangle \in H$;

(4) two binary relation symbols E and S.

We now let T_0 be the theory in L_0 consisting of the following sentences:

(1) $U^i \cap U^j = 0$ whenever $i < j < n$, and $\{U^0, \cdots, U^{n-1}\}$ is a partition;

(2) $U_i^k \subseteq U^k$ whenever $k < n$ and $i < \omega$;

(3) $U_i^k \cap U_j^k = 0$ whenever $k < n$ and $i < j < \omega$;

(4) $|U_i^k| = i + 1$ whenever $k < n$ and $i < \omega$;

(5) E is an equivalence relation;

(6) U_i^k is an equivalence class of E and U^k is the union of equivalence classes of E whenever $k < n$ and $i < \omega$;

(7) $S = 0$;

(8) $R_{cde}(x, y, z) \rightarrow (U^c(x) \wedge U^d(y) \wedge U^e(z))$, whenever $\langle c, d, e \rangle \in H$;

(9) $(R_{cde}(x, y, z) \wedge U_m^e(z)) \rightarrow \bigvee_{i \leqq m} (U_i^c(x) \vee U_i^d(y))$, whenever $\langle c, d, e \rangle \in H$ and $m < \omega$;

(10) $U^c(x) \wedge U^d(y) \wedge U^e(z) \wedge \bigwedge_{i<m} \neg U_i^e(z) \wedge \bigvee_{i \leqq m} (U_i^c(x) \vee U_i^d(y)) \rightarrow R_{cde}(x,y,z)$ whenever $\langle c, d, e \rangle \in H$ and $m < \omega$.

Sentences (9) and (10) are not so formidable as they might appear. To give a more informal account of their content, for any x let $\rho(x) = i$ if there are $k < n$ and $i < \omega$ such that $x \in U_i^k$. In case there are no such i and k, then let $\rho(x) = \infty$ with the usual conventions concerning $<$.

Now suppose $x \in U^c$, $y \in U^d$ and $z \in U^e$. Then sentences (9) and (10) together assert that

$$R_{cde}(x, y, z) \Leftrightarrow \min(\rho(x), \rho(y)) \leqq \rho(z)$$

as best as can be asserted with a set of first-order sentences.

Notice that the theory T_0 is *positively inductive*; that is, the direct limit of a set of models of T_0 directed under positive embeddings is also a model of T_0. Thus, a p.e.c. structure is a model of T_0. In this Theorem we actually require that T_0 be negative universal, and it is not. But there is no problem because the negative universal part of T_0 can be considered just as well.

From sentences (1)–(10) it follows that there is a unique model $\mathfrak{A}$ of T_0 such that $\rho(a) < \infty$ for each $a \in A$. This model is p.e.c. and embeddable in any other model of T_0. Conversely, no other model is positively embeddable in $\mathfrak{A}$.

The following facts concerning models $\mathfrak{A}$ and $\mathfrak{B}$ of T_0 are all rather evident.

1. If $\mathfrak{A} \subseteq_p \mathfrak{B}$ and $x \in A$, then $\rho^{\mathfrak{A}}(x) = \rho^{\mathfrak{B}}(x)$. (Thus, we can unambiguously refer to $\rho(x)$.)

2. If $\mathfrak{A} \models (U^k(x) \wedge U^k(y))$ for some $x, y \in A$ and $\rho(x) = \rho(y)$, then there is a model $\mathfrak{A}_1$ of T_0 such that $\mathfrak{A} \subseteq_p \mathfrak{A}_1$ and $\mathfrak{A}_1 \models E(x, y)$.

3. If $x, y, z \in A$ are such that $\mathfrak{A} \models U^c(x) \wedge U^d(y) \wedge U^e(z)$ and $\min(\rho(x), \rho(y)) \leqq \rho(z)$, then there is a model $\mathfrak{A}_1$ of T_0 such that $\mathfrak{A} \subseteq_p \mathfrak{A}_1$ and $\mathfrak{A}_1 \models R_{cde}(x, y, z)$.

From facts 1–3 above we can make the following observation. Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are models of T_0, and that $\mathfrak{B}$ is p.e.c. Let $f : A \to B$ be a one-one function such that whenever $x \in A$, $i < \omega$ and $k < n$, then $\mathfrak{A} \models U^k_i(x)$ iff $\mathfrak{B} \models U^k_i(f(x))$, and $\mathfrak{A} \models U^k(x)$ iff $\mathfrak{B} \models U^k(f(x))$. Then $f : \mathfrak{A} \to \mathfrak{B}$ is even a positive embedding. If, in addition, $\mathfrak{A}$ is p.e.c., then f is an embedding. Furthermore, if f is a bijection, then f is an isomorphism.

For a model $\mathfrak{A}$ of T_0, let $V(\mathfrak{A}) = \{k < n : \mathfrak{A} \models \exists x(U^k(x) \wedge \rho(x) = \infty)\}$. It is clear that if $\mathfrak{A}$ is p.e.c. and $k \in V(\mathfrak{A})$, then $\{x \in A : \mathfrak{A} \models (U^k(x) \wedge \rho(x) = \infty)\}$ is infinite. Thus, from the above observations, we can easily conclude that if $\mathfrak{A}$ and $\mathfrak{B}$ are countable p.e.c. models, then $\mathfrak{A}$ is embeddable in $\mathfrak{B}$ if $V(\mathfrak{A}) \subseteq V(\mathfrak{B})$, and also $\mathfrak{A}$ is isomorphic to $\mathfrak{B}$ iff $V(\mathfrak{A}) = V(\mathfrak{B})$.

Because of the inclusion of sentences (8) and (9) in the theory T_0, it easily follows that if $\mathfrak{A}$ is a p.e.c. model of T_0 which is not the minimal one, then there is some $k < n$ such that $V(\mathfrak{A}) = \{j < n : p_j \leqq p_k\}$. For, by way of contradiction, suppose that $c, d \in V(\mathfrak{A})$ are such that p_c and p_d are maximal, incomparable elements of $\{p_j : j \in V(\mathfrak{A})\}$. Then there is $e \notin V(\mathfrak{A})$ such that $\langle c, d, e \rangle \in H$. If a_0, a_1 are such that $\rho(a_0) = \rho(a_1) = \infty$ and $\mathfrak{A} \models (U^c(a_0) \wedge U^d(a_1))$, then due to $\mathfrak{A}$ being p.e.c. and fact (3), there is b such that $\mathfrak{A} \models R_{cde}(a_0, a_1, b)$. By sentence (9) $\rho(b) = \infty$. But then $e \in V(\mathfrak{A})$, which is a contradiction.

Conversely, it is easy to construct, for each $k < n$, a countable p.e.c. model $\mathfrak{A}$ of T_0 such that $V(\mathfrak{A}) = \{j < n : p_j \leqq p_k\}$. To do so, we first construct $\mathfrak{A}$ such that $V(\mathfrak{A}) = \{0, 1, \cdots, n-1\}$. Let $U^0, U^1, \cdots, U^{n-1}$ be countably infinite, pairwise disjoint sets, and set

$$A = \bigcup_{k<n} U^k.$$

For each $k < n$ and $i < \omega$ let $U_i^k \subseteq U^k$ such that

$$|U_i^k| = i + 1;$$

$$i < j < \omega \Rightarrow U_i^k \cap U_j^k = 0;$$

$$U_\omega^k = U^k - \bigcup_{i<\omega} U_i^k \quad \text{is infinite}.$$

Let E be the equivalence relation on A whose equivalence classes are just the U_i^k for $k < n$ and $i \leqq \omega$. For $\langle c, d, e\rangle \in H$, let $\langle x, y, z\rangle \in R_{cde}$ iff $x \in U^c$, $y \in U^d$, $z \in U^e$ and $\min(\rho(x), \rho(y)) \leqq \rho(z)$. Finally, set $S = 0$. We then let

$$\mathfrak{A} = (A, U_i^k, U^k, E, S, R_{cde})_{k<n, i<\omega, \langle c,d,e\rangle \in H}.$$

Now for each $k < n$, let $\mathfrak{A}_k \subseteq \mathfrak{A}$ where $A_k = \{x \in A : \text{if } x \in U_\omega^j, \text{ then } p_j \leqq p_k\}$. Clearly $\mathfrak{A}_k$ is a model of T_0 and $V(\mathfrak{A}_k) = \{j < n : p_j \leqq p_k\}$. We need only show that each $\mathfrak{A}_k$ is p.e.c. This is quite routine, and is left as an exercise.

Finally, notice that T_0 has PJEP. For, if $\mathfrak{A}$ is p.e.c. and $V(\mathfrak{A}) = \{0, 1, \cdots, n-1\}$, then every countable $\mathfrak{B}$ is positively embeddable in $\mathfrak{A}$.

This proves the Theorem (positive version) in the case $Q = 0$. We indicate the changes to make for arbitrary $Q \subseteq P$. Without loss of generality we will assume that $p_n \not\in Q$. (For, notice that Q-equivalence is the same as $(Q \cup \{p_n\})$-equivalence.) To define T_Q, in the definition of T_0 replace sentence (7) with

(7_Q) S is a reflexive, partial order defined on U^k such that $S(x, y) \Rightarrow E(x, y)$ and $S(x, x) \Rightarrow x \in U^k$ for some $p_k \in Q$.

If $\mathfrak{A}$ is a p.e.c. model of T_Q, $p_k \in Q$ and $\mathfrak{A} \models U^k(a)$, then S linearly orders $\{b \in A : \mathfrak{A} \models E(a, b)\}$. Furthermore, if $\rho(a) = \infty$ and $\mathfrak{A}$ is countable, then the order type is that of the rationals. Thus, if $\mathfrak{A}$ is p.e.c. and $p_k \in Q$, then $\mathfrak{A}$ is a model of the sentence "there is an E-equivalence class which is densely ordered by S" holds in $\mathfrak{A}$ iff $k \in V(\mathfrak{A})$.

This finishes the proof of the Theorem (positive version).

To prove the Theorem we will modify the theories T_Q. Let L be the language consisting of the previously used symbols U_i^k and U_i together with the following symbols:

(1) unary relation symbols U, J_1, J_2, and I_{cde} for each $\langle c, d, e\rangle \in H$;

(2) ternary relation symbols E' and S';

(3) 4-ary realtion symbols R'_{cde} for each $\langle c, d, e\rangle \in H$.

For each L_0-structure $\mathfrak{A}$, we define an L-structure $\mathfrak{A}^*$ in the following way. Let

$$J_1 = \{\langle x, y, 1\rangle : \mathfrak{A} \models E(x, y)\},$$

$$J_2 = \{\langle x, y, 2\rangle : \mathfrak{A} \models S(x, y)\},$$

$$I_{cde} = \{\langle x, y, z, \langle c, d, e\rangle\rangle : \mathfrak{A} \models R_{cde}(x, y, z)\},$$

and

$$U = A.$$

Then set $A^* = J_1 \cup J_2 \cup \bigcup \{I_{cde} : \langle c, d, e\rangle \in H\} \cup U$. Let

$$E' = \{\langle x, y, \langle x, y, 1\rangle\rangle : \mathfrak{A} \models E(x, y)\},$$

$$S' = \{\langle x, y, \langle x, y, 2\rangle\rangle : \mathfrak{A} \models S(x, y)\}$$

and

$$R'_{cde} = \{\langle x, y, z, \langle x, y, z, \langle c, d, e\rangle\rangle\rangle : \mathfrak{A} \models R_{cde}(x, y, z)\}.$$

Then the structure $\mathfrak{A}^*$ is

$$(A^*, U_i^k, U^k, U, J_1, J_2, E', S', R'_{cde})_{k<n,\, i<\omega,\, \langle c,d,e\rangle \in H}.$$

Now let

$$T^* = \mathrm{Th}(\{\mathfrak{A}^* : \mathfrak{A} \text{ is a model of } T_Q\}).$$

For a model $\mathfrak{B}$ of T^* there is a unique $\mathfrak{A}$ (to within isomorphism) which is a model of T_Q and for which $\mathfrak{A}^* \cong \mathfrak{B}$. The following facts are easily ascertained for models $\mathfrak{A}$ and $\mathfrak{B}$ of T_Q.

1. $\mathfrak{A}$ is countable iff $\mathfrak{A}^*$ is countable.
2. If $f : \mathfrak{A} \to \mathfrak{B}$ is a positive embedding, then there is a unique embedding $f^* : \mathfrak{A}^* \to \mathfrak{B}^*$ such that $f^* \mid A = f$.
3. If $g : \mathfrak{A}^* \to \mathfrak{B}^*$ is an embedding, then $(g \mid A) : \mathfrak{A} \to \mathfrak{B}$ is a positive embedding.
4. $\mathfrak{A}$ is p.e.c. iff $\mathfrak{A}^*$ is an existentially complete model of T^*.
5. $\mathfrak{A} \equiv \mathfrak{B}$ iff $\mathfrak{A}^* \equiv \mathfrak{B}^*$.

From 2 it is clear that T^* has JEP. From the above facts it easily follows that if we let T be the universal part of T^*, then T is the desired example. □

The theory T constructed in the proof of the Theorem was in an infinite language. To convert the example to one in a finite language, we make use of an example of Peretyatkin [2]. He showed that in the language consisting of one binary relation symbol there is a theory H and a sequence $\langle \sigma_n : n < \omega\rangle$ of sentences with the following properties. If $X \subseteq \omega$, then let $T_X = H \cup \{\sigma_n : n \in X\} \cup \{\neg \sigma_n : n \notin X\}$.

1. Each T_X is complete, consistent, model-complete and $\aleph_0$-categorical.

2. If $X_i \subseteq \omega$ and $\mathfrak{B}_i$ is a countable model of T_{X_i} $(i = 1, 2)$, then $\mathfrak{B}_1$ is embeddable in $\mathfrak{B}_2$ iff $X_1 \subseteq X_2$.

For $n < \omega$ let $Y_n = \{2i : i \leqq n\} \cup \{2i + 1 : n < i < \omega\}$, and then let $Y_\omega = \lim_n Y_n = \{2i : i < \omega\}$. Notice that if $i, j \leqq \omega$ and $Y_i \subseteq Y_j$, then $i = j$. Let $\mathfrak{B}_n$ be a countable model of T_{Y_n}. Given a model $\mathfrak{A}$ of T, construct $\mathfrak{A}'$ by "attaching" to each $a \in U_i^k$ a copy of $\mathfrak{B}_i$, and then disgregard the relations U_i^k. Because of Properties 1 and 2 above and the definition of the Y_i, the universal part of the theory $T' = \mathrm{Th}(\{\mathfrak{A}' : \mathfrak{A} \text{ is a model of } T\})$ will have the same properties as T yet be in a finite language.

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